FIBONACCI MATRICES

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Problem: The determinant of the $n \times n$ matrix below is the nth Fibonacci number.

$$\begin{pmatrix} 1 & i & ... & ... & 0 \\ i & 1 & i & ... & ... \\ ... & i & 1 & i & ... \\ ... & ... & i & 1 & i \\ 0 & ... & ... & i & 1 \end{pmatrix}$$

Where all the ... are 0s.

Solution: We'll solve this problem by induction.

Call the nth matrix A_n.

For induction, we need to show that the equation is true for the first cases. By a simple calculation, we know that $\det A_1 = 1$ and $\det A_2 = 2$. These are F_1 and F_2 respectively, so we're set.

The second step of induction is to assume that $\det A_{n-1} = F_{n-1}$ and $\det A_n = F_n$.

The third step of induction asks us to show, using our assumptions, that $\det A_{n+1} = F_{n+1}$. Let's do a brute force calculation!

$$\det A_{n+1} = \det \begin{pmatrix} 1 & i & ... & ... & 0 \\ i & 1 & i & ... & ... \\ ... & i & 1 & i & ... \\ ... & ... & i & 1 & i \\ 0 & ... & ... & i & 1 \end{pmatrix}$$

$$\det A_{n+1} = 1 \det \begin{pmatrix} 1 & i & ... & 0 \\ i & 1 & i & ... \\ ... & i & 1 & i \\ 1 & ... & i & 1 \end{pmatrix} - i \det \begin{pmatrix} i & i & ... & 0 \\ ... & 1 & i & ... \\ ... & i & 1 & i \\ 0 & ... & i & 1 \end{pmatrix}$$

We see that the first term is simply $\det A_n$. What we want to do is show that the second term is $\det A_{n-1}$. This is easily seen if we simplify the determinant by taking the determinant of the transposition of the matrix. (A basic linear algebra fact is $\det M = \det M^T$.)

$$\det \begin{pmatrix} i & i & ... & 0 \\ ... & 1 & i & ... \\ ... & i & 1 & i \\ 0 & ... & i & 1 \end{pmatrix}^T = \det \begin{pmatrix} i & ... & ... & 0 \\ i & 1 & i & ... \\ ... & i & 1 & i \\ 0 & ... & i & 1 \end{pmatrix}$$

Since the top row of this matrix is all 0s, we can quickly see that the determinant is $i \det A_{n-1}$.

Now we're done. Rewriting the following equation

$$\det A_{n+1} = 1 \det \begin{pmatrix} 1 & i & ... & 0 \\ i & 1 & i & ... \\ ... & i & 1 & i \\ 1 & ... & i & 1 \end{pmatrix} - i \det \begin{pmatrix} i & i & ... & 0 \\ ... & 1 & i & ... \\ ... & i & 1 & i \\ 0 & ... & i & 1 \end{pmatrix}$$

we get

$$\det A_{n+1} = 1 \det A_n - i(i \det A_{n-1}) = \det A_n + \det A_{n-1} = F_n + F_{n-1} = F_{n+1}$$

This is what we hoped to show.