Pile of Balls

Balls of radius r are stacked in a square pyramid with n layers. Find the volume of the enclosing square pyramid.

Geometric derivation

Let ABCD denote the centers of the four balls at the vertices of the base and let T be the center of the top ball. Then, since the sides of the pyramid ABCDT are congruent equilateral triangles, all the edges of this pyramid have the same length s where

$$s = 2(n-1)r.$$

Let O be the center of the square ABCD and M the midpoint of one of the sides AB. Triangle AMT has a right angle at M, hypotenuse $\text{TA} = s$ and base $\text{AM} = s/2$ so the height of the triangular sides is, by Pythagoras, $\text{TM} = (\sqrt{3}/2)s$. This is the hypotenuse of the right triangle OTM which has base $\text{MO} = s/2$ so the height of the pyramid $\text{OT} = (\sqrt{2}/2)s$, again by Pythagoras.

Below is a side view of the slice through the middle of the pyramid. M, T and O. The primed points indicate the enclosing pyramid. The height h of the enclosing pyramid is

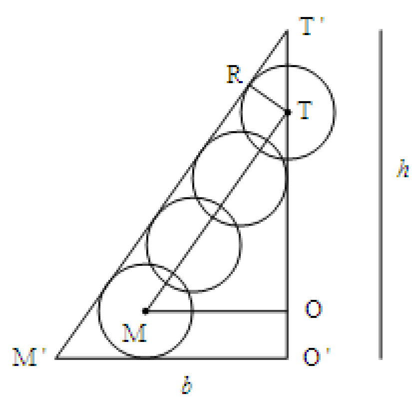
$$h = \text{O'T'} = \text{O'O} + \text{OT} + \text{TT'} = r + (\sqrt{2}/2)s + \sqrt{3}\,r = \left(1 + \sqrt{2}(n-1) + \sqrt{3}\right)r$$

since, by similar triangles OTM and RT'T,

$$\frac{\text{TT'}}{\text{TR}} = \frac{\text{TT'}}{r} = \frac{\text{TM}}{\text{MO}} = \frac{(\sqrt{3}/2)s}{s/2} = \sqrt{3} \quad \text{so} \quad \text{TT'} = \sqrt{3}\,r.$$

The square base has half-side length $b = h/\sqrt{2}$ so the volume of the enclosing pyramid is

$$\begin{aligned} V &= \frac{1}{3}(\text{area of base})(\text{height}) \\ &= \frac{1}{3}(2b)^2 h = \frac{1}{3}(\sqrt{2}h)^2 h = \frac{2}{3}\left(1 + \sqrt{2}(n-1) + \sqrt{3}\right)^3 r^3. \end{aligned}$$



Analytic geometry

Using the same labels as above, put the origin at O and the x, y and z axes through A, B and T respectively. Consider the plane passing through the intercepts A, B and T. Since the distance between these points is $2(n-1)r$, each lies a distance $2(n-1)r/\sqrt{2}$ from the origin, so the plane has equation

$$x+y+z=\sqrt{2}(n-1)r.$$

Translating this plane in the direction of its normal vector $\langle 1,1,1\rangle$ changes the equation to

$$x+y+z=\sqrt{2}(n-1)r+\sqrt{3}\,r,$$

the $\sqrt{3}$ being the length of the normal. This plane is now tangent to one face of the pyramid of balls. We need to move it an additional distance r in the z direction to keep it tangent to this face when we lift the entire structure so it sits on the xy plane ...

$$P: x+y+z=r+\sqrt{2}(n-1)r+\sqrt{3}\,r=h \text{ say.}$$

Plane P now lies along one of the sides of the pyramid and the xy plane lies along the bottom. The enclosing pyramid has height h and square base with sides of length $\sqrt{2}h$ (to see this just look at the intercepts of P) so the volume is

$$\begin{aligned} V &= \frac{1}{3}(\text{area of base})(\text{height}) \\ &= \frac{1}{3}(\sqrt{2}h)^2 h = \frac{2}{3}\Big(1+\sqrt{2}(n-1)+\sqrt{3}\Big)^3 r^3. \end{aligned}$$

The question posed has $n=4$ so

$$V=\frac{2}{3}\Big(1+3\sqrt{2}+\sqrt{3}\Big)^3 r^3.$$